RINGS WHOSE PROPER IMAGES ARE ALMOST SELF-INJECTIVE

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Abstract. Motivated by a property of Dedekid domain, L. Levy (Pacific J. Math. **18**(1), (1966), 149-153.) characterized commutative noetherian rings whose proper homomorphic images are self-injective. The purpose of this paper is to characterize commutative noetherian rings whose proper homomorphic images are almost self-injective, a property that holds for a larger class of rings, including serial rings.

Introduction

The following theorem was proved by L. Levy in [6].

Theorem 1. *Let R be a commutative noetherian ring. Then every proper homomorphic image of R is self-injective if and only if R is one of the following types of rings (I) Principal ideal artinian ring, (II) Dedekind domain, (III) Local ring whose maximal ideal M has composition length 2 and satisfies $M^2 = 0$.*

The main result of our paper is the following theorem:

Theorem 2. *Let R be a commutative noetherian ring. Then every proper homomorphic image of R is almost self-injective if and only if R is a direct sum of rings of the following three types of rings (not necessarilly all):*
(I) Serial ring, (II) Dedekind domain, (III) Local ring with the maximal ideal $M = A \oplus B = Soc(R)$. Furthermore, in each case the Krull dimension is bounded by 2.

The concept of almost self-injective ring was introduced by Baba in [2] and studied, among others, by Harada and Tozaki [5] in connection with serial rings and almost quasi-Frobenious rings.

A commutative ring R is almost self-injective, if for any R-homomorphism $f : I \longrightarrow R$, I an ideal of R, either f extends to a homomorphism $g : R \longrightarrow R$, or there exists a decomposition $R = R_1 \oplus R_2$ and a homomorphism $h : R \longrightarrow R_1$, where $R_1 \neq 0$, such that $hf(x) = \pi(x)$ for all $x \in I$, π is the usual projection of R onto R_1. In other words one of the following two diagrams can be completed:

$$(1)\quad \begin{tikzcd} 0 \arrow[r] & I \arrow[r, "i"] \arrow[d, "f"'] & R \arrow[dl, "g"] \\ & R & \end{tikzcd} \qquad (2)\quad \begin{tikzcd} 0 \arrow[r] & I \arrow[r, "i"] \arrow[d, "f"'] & R = R_1 \oplus R_2 \arrow[d, "\pi"] \\ & R \arrow[r, "h"] & R_1 \end{tikzcd}$$

and we will sometimes say that the map f can be "completed" (by g or h).

Indeed the concept of almost injectivity has defined in a similar way for any module over a ring with identity (not necessarily commutative, cf. [2], [5]) .

R is called a uniserial ring or a valuation ring if each pair of ideals (equivalently principal ideals) can be compared by the inclusion relation. A ring R is called serial if it is a direct sum of uniserial rings.

All rings considered in this paper are commutative and have identity different from zero.

1. Main result

Let us first mention that an almost self-injective ring which is indecomposable must be uniform. Indeed such a ring has no idempotents and hence it is π-injective (=quasi-continuous) (cf. Lemma 1, [1]). Moreover, an almost self-injective noetherian ring is a finite direct sum of uniform ideals (cf. Proposition 1.10 in [7]).

It is easy to prove that the direct sum of almost self-injective rings is almost self-injective.

We start by showing that a finite direct sum of valuation rings is such that every homomorphic image is almost self-injective.

Lemma 3. *Every homomorphic image of a finite direct sum of noetherian valuation rings is almost self-injective.*

Proof. Let us first consider the case of a single valuation ring. It is obvious that each homomorphic image of a valuation ring is still a valuation ring. So, it is enough to show that a valuation ring is an almost self-injective ring. Notice also that a noetherian valuation ring is in fact a local PID and hence also indecomposable. Let $f : aR \longrightarrow R$ be an R-module homomorphism. Let $f(a) = b$. Consider the case when $aR \subseteq bR$. Then there exists $c \in R$ such that $a = bc$. We define $h \in End_R(R)$ by $h(1) = c$. Then $h(f(ar)) = h(br) = brc = ar$, for any $r \in R$ and thus $h \circ f = \mathrm{Id}_{aR}$. On the other hand if there exists $d \in R$ such that $b = ad$, we define $g \in End_R(R)$ by $g(1) = d$. Then $g(a) = ag(1) = ad = b = f(a)$. Thus in this case g extends f to R. The proof is easily completed since a finite direct sums of almost self-injective rings is almost self-injective.

□

Lemma 4. *In an almost self-injective indecomposable ring if two elements $a, b \in R$ are such that $ax = 0$ implies $bx = 0$, then either $Ra \subset Rb$ or $Rb \subset Ra$. In particular, if R is a domain then it is a valuation domain.*

Proof. Since the map $Ra \to Rb$ sending a to b is well defined and can be completed in one of the two ways as given in the definition of almost self-injectivity. These two different ways lead to the fact that either a divides b or b divides a. □

Our next proposition will be useful in proving our main result.

Proposition 5. *Let R be a commutative noetherian ring having the property that every proper homomorphic image is almost self-injective. Then the following hold:*

(1) *If R is a domain then it is a Dedekind domain and the Krull dimension of R is 1. Moreover, if R is istelf almost self-injective then R is a valuation ring with a unique nonzero prime ideal (= maximal ideal) and all the ideals are power of this prime ideal.*

(2) *If R is not a domain, there exists a finite number of maximal ideals and the Krull dimension of R is bounded by 2.*

Proof. (1) We first claim that there are no ideals between M and M^2, where M is a maximal ideal. Since R is a domain, $M^2 \neq 0$ and the ring $S = R/M^2$ is a local almost self-injective ring. If x, y are nonzero elements in M/M^2 then x and y have the same annihilator, namely M/M^2. This implies that $xS \cong yS$. Since S is almost self-injective the map $\varphi : xS \longrightarrow S$ given by $\varphi(x) = y$ can be completed and hence either there exists $u \in S$ such that $y = xu$ or there exists $v \in S$ such that $x = yv$. Since the square of M/M^2 in $S = R/M^2$ is zero, the elements $u, v \in S$ cannot belong to M/M^2. Thus u, v are invertible. This leads to the fact that for every pair of elements $x, y \in S$ we have $Sx = Sy$. This yields the claim. We can then use a theorem of Cohen (cf. [3]) that a commutative noetherian domain satisfying the property that there are no ideals between a maximal ideal M and its square must be a Dedekind domain.

If the ring R is itself almost self-injective Lemma 4 above shows that the ring R is a valuation ring. Since in a Dedekind domain every ideal is a product of prime ideals and every nonzero prime ideal is maximal the last statement follows.

(2) It is well known that a commutative noetherian ring has only a finite number of minimal prime ideals. If P is such a minimal prime ideal, then R/P is a domain and Lemma 4 shows that it is a valuation domain and hence a local ring. In particular, there is a unique maximal ideal containing P. This shows that the number of maximal ideals in R is fewer then the number of minimal prime ideals. Moreover the above statement (1) shows that every ideal in R/P is a power of the maximal ideal. In particular, the Krull dimension of R is bounded by 2.

□

Before proving the main theorem we prove the following crucial result for a local ring satisfying our property i.e. every proper homomorphic image is almost self-injective.

Lemma 6. *Let R be a commutative local ring with maximal ideal M such that each proper homomorohic image is almost self-injective. Then either R is a valuation ring or $M^2 = 0$ with composition length of M equals to 2. Indeed, every proper homomorphic image is a valuation ring in each case.*

Proof. We claim that every ideal is either minimal or essential. Suppose there exist nozero ideals I and K such that I is not minimal and $I \cap K = 0$. Then for every nonzero ideal C properly contained in I, $(I/C) \cap (K + C)/C = 0$. Since R is local and $C \neq 0$, R/C is local and almost self-injective, and hence uniform. In particular, I/C is essential, a contradiction. This proves the claim. Since every proper homomorphic image of R is uniform, $udim(R) \leq 2$. In particular, the socle of R is a direct sum of at most two minimal ideals. We divide the proof in three cases.

Let the socle of R be zero. Then every ideal of R is essential. Let I and K be two nonzero ideals. Since $R/(I \cap K)$ is uniform and $\frac{I}{I \cap K} \cap \frac{K}{I \cap K} = 0$, it follows that $I \subseteq K$ or $K \subseteq I$. Thus R is a valuation ring.

If the socle or R consists of a single minimal ideal, then all ideals are essential and the proof comes from the above case.

Finally, let $soc(R) = A \oplus B$ where A and B are minimal ideals. Let M be the unique maximal ideal. Then $MSoc(R) = 0$. In particular, for any $x \in M$ we have that $ann(x) \neq 0$ and $Rx \cong R/ann(x)$ is uniform. This means that Rx cannot contain the $Soc(R) = A \oplus B$ and hence Rx cannot be essential. So Rx must be minimal. This shows that $M \subseteq Soc(R)$ and so $M = Soc(R)$. Clearly, $M^2 = 0$, as desired.

Furthermore every proper homomorphic image of R is clearly a valuation ring, completing the proof. $\square$

We now prove the main Theorem 2 as stated in the introduction.

Proof. We denote the prime radical of R by N.

Suppose first that $N \neq 0$. Then all homomorphic images of R/N are almost self-injective. Invoking the result (8.2, p.66, [4]) , we get $R/N = \oplus_{i=1}^{l} e_i R/e_i N$ where $\{e_1 + N,...,e_k + N\}$ is an orthogonal family of idempotents such that $1 + N = (e_1 + N) + ... + (e_k + N)$. Since N is nilpotent, without loss of generality we can assume that e_i are idempotents in R (cf. Theorem 21.28 p.319 [8]), and thus the decomposition of R/N can be lifted to $R = \oplus_{i=1}^{l} e_i R$ say.

If $l > 1$, then each $e_i R$ is uniform almost selfinjective and hence a local ring by Theorem 5 in [1]. In this case Lemma 6 finishes the proof. Now suppose $l = 1$. As above we conclude that R/N is a local ring, this implies R that R is local. The conclusion follows by the above lemma 6. This completes the proof in the case when $N \neq 0$.

Suppose now that $N = 0$. Then R is a semiprime noetherian ring. We prove the result by induction on the uniform dimension of R. If $u.dim(R) = 1$, then R is domain and hence a Dedekind domain by the Lemma 4

Let us suppose that the result holds for commutative noetherian rings with $udim(R) < n$, for some $n > 1$ We consider two cases. First suppose, there exists an non-essential maximal ideal M. Then $R = M \oplus K$ for some nonzero ideal K. Then by the chinese remainder theorem $R \cong R/M \times R/K$, where R/M is a field. Since $udim(R/K) < udim(R)$, the induction hypothesis gives the desired conclusion.

If on the other hand every maximal ideal is essential, then according to Proposition 5 the minimal prime ideals are also essential. Since there are only a finite number of minimal prime ideals the nilpotent radical N is essential, a contradiction because R is semiprime. This proves the "if" part of the theorem..

"only if part". Let us now show that if a ring R belongs to one of the three families given in the statement of the main theorem then every proper homomorphic image of R is almost self-injective. If R is a Dedekind domain then it is wellknown that every proper image of R is self injective. For the remaining two types, refer to Lemmas 6 and 3. $\square$

We may compare our result with the one obtained by L. Levy in [6]. Firstly, we remark that a principal ideal artinian ring is also a valuation ring. The example given below shows that there exists a family of rings of a type obtained by us which does not fall in the class of rings obtained by Levy.

Example 7. *Let k be a field and consider the product $R = k[[X]] \times k[[Y]]$. This ring is not a domain. It is neither a local ring nor artinian but it is the product of two valuation domains and hence satisfies our property. So this ring doesn't belong to the family obtained by Levy but belongs to the third family we obtained. In other words this ring is restricted almost self-injective but not restricted self-injective.*

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